On the evaluation of the improvement parameter in the lattice Hamiltonian approach to critical phenomena

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Abstract

In lattice Hamiltonian systems with a quartic coupling γ, a critical value γ^* may exist such that, when $\gamma = \gamma^*$, the leading irrelevant operator decouples from the Hamiltonian and the leading nonscaling contribution to renormalization-group invariant physical quantities (evaluated in the critical region) vanishes. The $1/N$ expansion technique is applied to the evaluation of γ^* for the lattice Hamiltonian of vector spin models with $O(N)$ symmetry. As a byproduct, systematic asymptotic expansions for the relevant lattice massive one-loop integrals are obtained.

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I. INTRODUCTION

The quest for better analytical and numerical methods in the theoretical evaluation of measurable physical quantities, like critical exponents and amplitude ratios, is one of the lasting tasks of statistical field theory.

In recent years substantial progress in this field has been made by the introduction of a method based on the strong coupling lattice expansion of improved Hamiltonians [1] (for a review, cf. Ref. [2]). The essential feature of this method is the possibility of removing all leading nonscaling contributions to physical quantities, in the neighborhood of criticality, by a specific choice of a parameter in the lattice Hamiltonian (critical coupling). The convergence of analytical and/or numerical evaluations is therefore impressively faster than in any other variant form of the models under inspection belonging to the same universality class.

The main limitation of this method lays in the absence of an efficient analytical technique for the determination of the critical parameter. Conceptual reasons for this limitation may be found in the impossibility of an ϵ expansion for the value of the critical parameter, both in the $4-\epsilon$ and in the $2+\epsilon$ expansion schemes. In practice one must resort to an extrapolation from numerical Monte Carlo finite-size evaluations of some physical quantity, typically the Binder cumulant [1, 2].

Another well-known analytical approach to the study of critical lattice models is the $1/N$ expansion, which applies in particular to the physically very important class of $O(N)$ spin models in three dimensions. However it is known that, in exactly three dimensions and for nearest-neighbor interactions, the critical parameter in the large-N limit is a negative number [3]. Since the critical parameter controls the large-field behavior of the interaction potential, a negative value would naively imply an unbounded Hamiltonian. In practice this would prevent a Monte Carlo simulation of the system, thus seriously jeopardizing the conceptual meaning of the whole approach.

Nevertheless we still seriously believe the $1/N$ expansion to be, from a theoretical point of view, probably the most relevant expansion scheme that can be applied to any quantum and statistical field theory, in that there is no known obstruction to summability of the series expansion in powers of $1/N$ for the values of physical quantities.

Therefore we decided to explore the conceptual and numerical consequences of performing

a systematic $1/N$ expansion of the critical parameter, for the class of three-dimensional $O(N)$ spin models, in order to check the actual relevance of the drawbacks that we mentioned above.

As a consequence of our analysis we found that the critical parameter can be formally computed within the expansion with no limitation related to the sign of its large-N value and for space dimensionalities in the interval $2 < d < 4$. In particular we found that the sign of the first $1/N$ correction is positive, and one may then expect to find a value N_c such that the parameter itself vanishes. It is however still unclear that the predictions of the $1/N$ expansion may be extended to the region $N < N_c$.

As a byproduct of our analysis we obtained new more efficient expressions for the asymptotic expansions of many important functions entering our calculations. We presented these results with some details because they might be relevant to other computations of critical and subcritical quantities.

In Sec. II we introduce the lattice $O(N)$ models and their $1/N$ expansion. In Secs. III, IV, and V we discuss on general grounds the relevant asymptotic epansions (gap equation, effective propagator and renomalized coupling). In Secs. VI and VII we specialize these expansions to the case of the standard nearest-neighbor interaction, with the help some useful integral representations. A few numerical results are presented in Sec. VIII. In Sec. IX we compute the $1/N$ correction to the improvement parameter and finally in Sec. X we discuss the meaning and relevance of our results.

II. THE EFFECTIVE HAMILTONIAN AND THE GRAPH EXPANSION

Our starting point will be the usual N-component ϕ^4 lattice Hamiltonian in d dimensions:

$$H = \sum_x \left[\frac{1}{2} \sum_\mu \nabla_\mu \boldsymbol{\phi}(x) \cdot \nabla_\mu \boldsymbol{\phi}(x) + \frac{1}{2} \mu_0^2 \boldsymbol{\phi}^2(x) + \frac{1}{4!} g_0 (\boldsymbol{\phi}^2(x))^2 \right], \tag{1}$$

where $\nabla_\mu \boldsymbol{\phi}(x)$ is some (local) form of the lattice gradient; in the standard nearest-neighbor formulation $\nabla_\mu \boldsymbol{\phi}(x) = \boldsymbol{\phi}(x+\mu) - \boldsymbol{\phi}(x)$.

Following Ref. [4] we define the rescaled couplings

$$\beta = -\frac{6\mu_0^2}{g_0 N}, \qquad \gamma = \frac{3}{g_0 N}$$

and introduce an auxiliary field α in order to eliminate the quartic term in the Hamiltonian.

After a trivial Gaussian integration the resulting effective Hamiltonian is

$$H_{\rm eff} = \frac{N}{2}[{\rm Tr\,Ln}\,\beta(-\nabla_\mu\nabla_\mu + i\alpha) - i\beta\alpha + \gamma\alpha^2]. \tag{2}$$

In the limit $\gamma \to 0$ this Hamiltonian reduces to the usual effective large-N expression for the nonlinear σ model. In the nearest-neighbor formulation, it is also known as the $O(N)$ Heisenberg model; its large-N limit was investigated in Ref. [5].

The saddle-point condition on the effective Hamiltonian leads to the so-called gap equation

$$\beta + 2\gamma m_0^2 = \int_{-\pi}^{\pi} \frac{d^dp}{(2\pi)^d}\frac{1}{\bar{p}^2 + m_0^2}, \tag{3}$$

where $\bar{p}^2$ is the Fourier transform of the lattice Laplacian operator $-\nabla_\mu\nabla_\mu$, which in the nearest-neighbor case takes the form $\hat{p}^2 = 2\sum_\mu(1-\cos p_\mu)$.

The gap equation allows for the elimination of β in favor of the new parameter m_0 (large-N inverse correlation length) in the Feynman graph expansion.

In the large-N limit criticality corresponds to the vanishing of m_0^2, and the criticality condition may then take the form

$$\beta_c = \int_{-\pi}^{\pi} \frac{d^dp}{(2\pi)^d}\frac{1}{\bar{p}^2},$$

corresponding to a finite value of β_c for all $d > 2$.

The graph expansion for this model, in the formulation based on the effective Hamiltonian, requires defining the (bare) propagator Δ for the effective field α; by standard manipulations one obtains

$$D(k, m_0, \gamma) \equiv \Delta^{-1}(k, m_0, \gamma) = \frac{1}{2}\int_{-\pi}^{\pi} \frac{d^dp}{(2\pi)^d}\frac{1}{\bar{p}^2 + m_0^2}\frac{1}{\overline{(p+k)}^2 + m_0^2} + \gamma. \tag{4}$$

The above approach is quite general, and it leads to a systematic $1/N$ expansion of the correlation functions and of the physical quantities for arbitrary values of m_0 and γ.

However we want to focus our attention on the critical domain, and in particular we want to evaluate the coupling γ^* such that the first nontrivial corrections to scaling turn out to vanish in the computation of physical quantities in the scaling region.

To this purpose it is convenient to parametrize the cutoff dependence of correlation functions and renormalized couplings in terms of the lattice spacing a, which can be made to appear explicitly in calculations by a rescaling of the coupling and momentum dependence.

Around criticality the dependence on a is not analytic, and as a consequence we need asymptotic expansions in order to identify the scaling and leading nonscaling contributions to any computable quantity.

The basic technique for asymptotic expansions in powers of $m_0 a$ is described in Ref. [6]; here we shall discuss its applications to the cases of interest for the present paper.

We only recall that, in order to regularize the generic lattice integral

$$I(k; m_0 a) \equiv \int_{-\pi}^{\pi} \frac{d^d p}{(2\pi)^d} F(k; m_0 a, p),$$

where k is any collection of external momenta, we can make use of the formal identity

$$I(k; m_0 a) = I_{\rm lat}(k; m_0 a) + I_{\rm con}(k; m_0 a),$$

where

$$I_{\rm lat}(k; m_0 a) = \int_{-\pi}^{\pi} \frac{d^d p}{(2\pi)^d} F(k; m_0 a, p) - \int_{-\infty}^{+\infty} a^d \frac{d^d p}{(2\pi)^d} T^{(\rm IR)} F(k; m_0 a, pa)$$

and

$$I_{\rm con}(k; m_0 a) = a^d \int_{-\infty}^{+\infty} \frac{d^d p}{(2\pi)^d} [F(k; m_0 a, pa) - T^{(\rm UV)} F(k; m_0 a, pa)].$$

The $T \equiv T^{(\rm IR)} + T^{(\rm UV)}$ operation amounts to a Taylor series expansion of the integrand in powers of $m_0 a$, $T^{(\rm IR)}$ and $T^{(\rm UV)}$ corresponding respectively to the IR and UV singular terms in the expansion.

It is possible to prove that the expansions of $I_{\rm lat}$ (the "lattice contribution") and $I_{\rm con}$ (the "continuum contribution") are individually and fully regular; the nonanaliticity of the expansion is factored out in the a^d term multiplying the continuum contribution.

III. ASYMPTOTIC EXPANSION OF THE GAP EQUATION

For the purposes of the present paper and in order to show an explicit example of the asymptotic expansion procedure let us consider the gap equation in the large-N limit.

Let us assume a generic lattice Laplacian such that $\bar{p}^2 \approx p^2 + c\, p^4 + O(p^6)$, where $p^{2n} = \sum_\mu p_\mu^{2n}$: in the nearest-neighbor version $c = -\frac{1}{12}$.

From the previously derived results we obtain the following relationship:

$$\beta_c - \beta = m_0^2 a^2 \left[\int_{-\pi}^{\pi} \frac{d^d p}{(2\pi)^d} \frac{1}{\bar{p}^2(\bar{p}^2 + m_0^2 a^2)} + 2\gamma \right].$$

Let us now define

$$G(p, m_0 a) \equiv \frac{m_0^2 a^2}{\bar{p}^2(\bar{p}^2 + m_0^2 a^2)}$$

and perform the relevant expansions up to the first few nontrivial terms:

$$\begin{aligned}
T\, G(p, m_0 a) &\approx \frac{m_0^2 a^2}{(\bar{p}^2)^2} - \frac{m_0^4 a^4}{(\bar{p}^2)^3} + O(m_0^6 a^6), \\
G(pa, m_0 a) &\approx \frac{1}{a^2}\left[\frac{1}{p^2} - \frac{1}{p^2 + m_0^2}\right] - c\, p^4 \left[\frac{1}{(p^2)^2} - \frac{1}{(p^2 + m_0^2)^2}\right] + O(a^2), \\
T\, G(pa, m_0 a) &\approx \frac{m_0^2}{a^2}\left[\frac{1}{(p^2)^2} - \frac{m_0^2}{(p^2)^3}\right] - m_0^2\, c\, p^4 \left[\frac{2}{(p^2)^3} - \frac{3 m_0^2}{(p^2)^4}\right] + O(m_0^6, a^2).
\end{aligned}$$

By grouping together the IR singular terms we therefore obtain the lattice contributions:

$$2 m_0^2 a^2 (\gamma - \gamma_0) - m_0^4 a^4 \delta_0 + O(m_0^6 a^6),$$

where we defined the following numerical constants:

$$\begin{aligned}
\gamma_0 &\equiv \frac{1}{2}\left[-\int_{-\pi}^{\pi} \frac{d^d p}{(2\pi)^d} \frac{1}{(\bar{p}^2)^2} + \int_{-\infty}^{+\infty} \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2)^2}\right], \\
\delta_0 &\equiv \int_{-\pi}^{\pi} \frac{d^d p}{(2\pi)^d} \frac{1}{(\bar{p}^2)^3} - \int_{-\infty}^{+\infty} \frac{d^d p}{(2\pi)^d} \left(\frac{1}{(p^2)^3} - 3c\, \frac{p^4}{(p^2)^4}\right).
\end{aligned} \tag{5}$$

On the other side, by grouping together the UV singular terms, we obtain the following continuum contributions:

$$\int_{-\infty}^{+\infty} \frac{d^d p}{(2\pi)^d} \left(a^{d-2}\left[-\frac{1}{p^2 + m_0^2} + \frac{1}{p^2}\right] + a^d c\, p^4 \left[\frac{1}{(p^2 + m_0^2)^2} - \frac{1}{(p^2)^2} - 2\frac{m_0^2}{(p^2)^3}\right] + O(a^{d+2})\right).$$

We can perform the continuum integrals by standard dimensional regularization techniques. We then finally find

$$\beta_c - \beta \approx -b_0\, (m_0 a)^{d-2} + 2(\gamma - \gamma_0) m_0^2 a^2 - \frac{3}{2} c\, b_0\, m_0^d a^d - \delta_0\, m_0^4 a^4 + O((m_0 a)^{d+2}), \tag{6}$$

where

$$b_0 \equiv \frac{\Gamma(1 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}}.$$

From the asymptotic expansion of the gap equation we immediately learn the following lesson: it is possible to choose for the quartic coupling γ the special value γ_0 such that the first nonscaling contribution to the large-N saddle-point condition vanishes.

By this procedure we have identified the large-N critical coupling for these versions of the model. In fact any change in the form of the local interaction, reflecting itself in the detailed

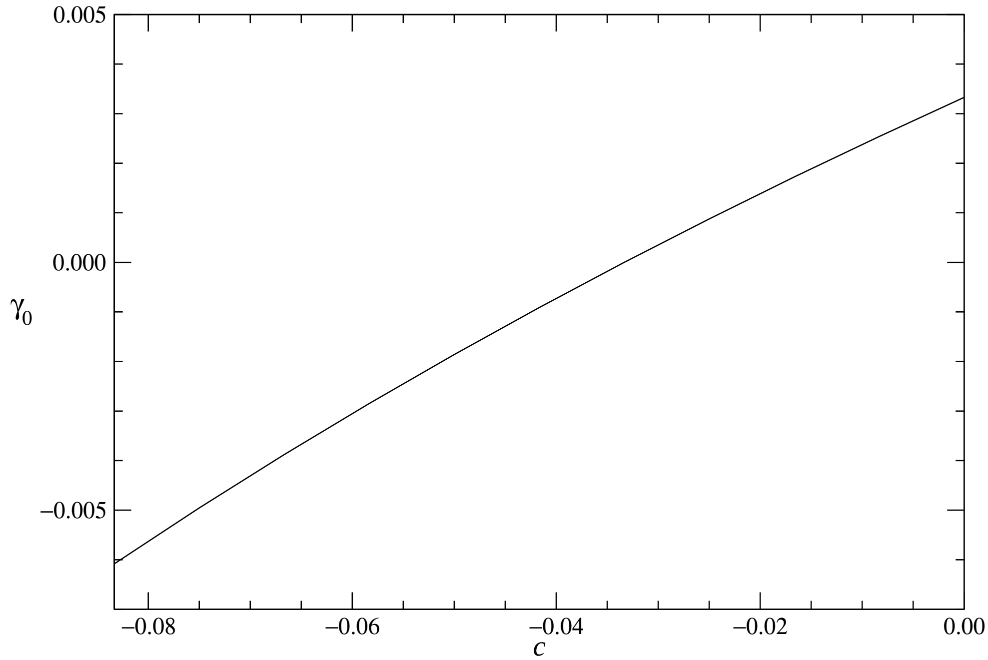

FIG. 1: γ_0 vs. c.

p dependence of the bare lattice massless propagator without changing its singular part, leads to a finite change in the numerical value of γ_0 without affecting its formal representation.

It is especially interesting to consider the class of models characterized by next-to-nearest neighbor interactions, and whose propagator of the fundamental excitations is obtained in the form

$$\bar{p}^2 = \sum_{\mu=1}^{d} (6c + \tfrac{5}{2}) - 8(c + \tfrac{1}{3})\cos p_\mu + 2(c + \tfrac{1}{12})\cos 2p_\mu = \hat{p}^2 + (c + \tfrac{1}{12})\hat{p}^4,$$

where

$$\hat{p}^n \equiv \sum_{\mu=1}^{d} \hat{p}_\mu^n, \qquad \hat{p}_\mu \equiv 2\sin\frac{p_\mu}{2}.$$

Note that $c = -\frac{1}{12}$ corresponds to the standard nearest-neighbor interaction, while $c = 0$ corresponds to the $O(a^2)$ Symanzik tree-improved version of $O(N)$ models [7].

In three dimensions we have numerically explored the range $-\frac{1}{12} \leq c \leq 0$: our results are presented in Figs. 1 and 2.

Let us notice in particular that the choice $c = -0.03332110...$ corresponds to vanishing γ_0, and it is therefore, at least in the large-N limit, an alternative version of a spin model where the leading corrections to scaling are automatically made to vanish. In turn when $c = 0$ we obtain $\gamma_0 = 0.003328210...$, which implies a small but nonvanishing leading scaling

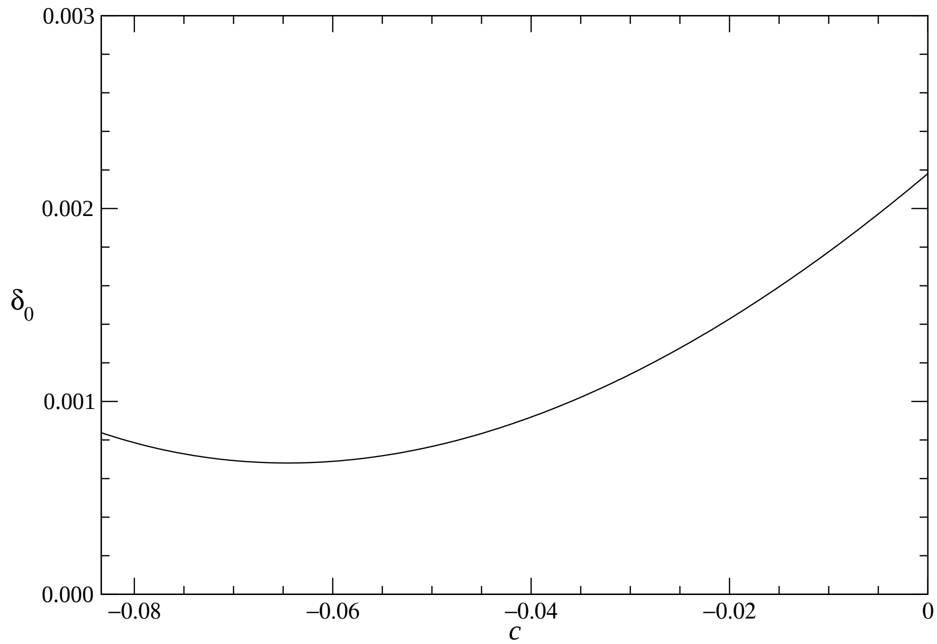

FIG. 2: δ_0 vs. c.

violation, and $\delta_0 = 0.002181406...$. Compared to the standard $c = -\frac{1}{12}$ case this version has however the advantage of being numerically testable also by Monte Carlo methods, since $\gamma_0 > 0$.

IV. ASYMPTOTIC EXPANSION OF THE EFFECTIVE PROPAGATOR

In order to perform an asymptotic expansion of the $O(1/N)$ contributions to physical quantities we must compute one-loop graphs involving the effective propagator $\Delta(k, m_0, \gamma)$.

We therefore need to evaluate the asymptotic expansion of Δ or, more conveniently, of $D(k, m_0, \gamma)$. It is easy to recognize, from the definition of D and from the general rule of the asymptotic expansion, that in general we may express the result in the form

$$D(k, m_0 a, \gamma) = \sum_{n=0}^{\infty} [A_n(k)(m_0 a)^{2n} + B_n(k)(m_0 a)^{2n+d-2}], \tag{7}$$

where $A_n(k)$ have the form of lattice contributions (and only A_0 depends on γ), while $B_n(k)$ are continuum contributions that can be analytically computed, e.g., in dimensional regularization.

We shall not repeat the derivation (some details can be found in Ref. [6]), but only quote

the relevant results:

$$\begin{aligned}
A_0(k,\gamma) &= \frac{1}{2}\int_{-\pi}^{\pi}\frac{d^dp}{(2\pi)^d}\frac{1}{\bar{p}^2}\frac{1}{\overline{(p+k)}^2}+\gamma,\\
A_1(k) &= -\int_{-\pi}^{\pi}\frac{d^dp}{(2\pi)^d}\frac{1}{(\bar{p}^2)^2}\frac{1}{\overline{(p+k)}^2}+\int_{-\infty}^{+\infty}\frac{d^dp}{(2\pi)^d}\frac{1}{(p^2)^2}\frac{1}{\bar{k}^2},\\
A_2(k) &= \int_{-\pi}^{\pi}\frac{d^dp}{(2\pi)^d}\frac{1}{(\bar{p}^2)^3}\frac{1}{\overline{(p+k)}^2}+\frac{1}{2}\frac{1}{(\bar{p}^2)^2}\frac{1}{((\overline{p+k})^2)^2}-\int_{-\infty}^{+\infty}\frac{d^dp}{(2\pi)^d}\frac{1}{(p^2)^3}\frac{1}{\bar{k}^2}\\
&\quad -3c\frac{p^4}{(p^2)^4}\frac{1}{\bar{k}^2}+\frac{1}{(p^2)^2}C_0(k)+\frac{1}{2}\left(\frac{1}{(p^2)^2}+\frac{1}{((p+k)^2)^2}\right)\frac{1}{(\bar{k}^2)^2};\\
B_0(k) &= \frac{b_0}{\bar{k}^2},\qquad B_1(k)=-b_0\left(\frac{1}{(\bar{k}^2)^2}+C_0(k)-\frac{3}{2}\frac{c}{\bar{k}^2}\right),
\end{aligned}\tag{8}$$

where we defined

$$C(k,m_0)\equiv\frac{1}{2d}\sum\nolimits_\mu\frac{\partial^2}{\partial k_\mu^2}\frac{1}{\bar{k}^2+m_0^2},\qquad C_0(k)\equiv C(k,0)\equiv\frac{1}{2d}\sum\nolimits_\mu\frac{\partial^2}{\partial k_\mu^2}\frac{1}{\bar{k}^2}.$$

Trivial manipulations allow to express $A_n(k)$ as pure lattice integrals:

$$\begin{aligned}
A_1(k) &= \int_{-\pi}^{\pi}\frac{d^dp}{(2\pi)^d}\frac{1}{(\bar{p}^2)^2}\left[\frac{1}{\bar{k}^2}-\frac{1}{\overline{(p+k)}^2}\right]+2\frac{\gamma_0}{\bar{k}^2},\\
A_2(k) &= \int_{-\pi}^{\pi}\frac{d^dp}{(2\pi)^d}\left[\frac{1}{(\bar{p}^2)^3}\frac{1}{\overline{(p+k)}^2}-\frac{1}{(\bar{p}^2)^3}\frac{1}{\bar{k}^2}-\frac{1}{(\bar{p}^2)^2}C_0(k)\right]+\frac{\delta_0}{\bar{k}^2}-2\gamma_0\,C_0(k)\\
&\quad+\int_{-\pi}^{\pi}\frac{d^dp}{(2\pi)^d}\left[\frac{1}{2}\frac{1}{(\bar{p}^2)^2}\frac{1}{((\overline{p+k})^2)^2}-\frac{1}{2}\frac{1}{(\bar{p}^2)^2}\frac{1}{(\bar{k}^2)^2}-\frac{1}{2}\frac{1}{((\overline{p+k})^2)^2}\frac{1}{(\bar{k}^2)^2}\right]-2\frac{\gamma_0}{(\bar{k}^2)^2}.
\end{aligned}\tag{9}$$

From our general considerations it should be by now clear that we shall also need a different expansion of D, homogeneous in powers of m and k.

Without delving into the details, we find that

$$D(ka,m_0a,\gamma)\approx\frac{1}{2}a^{d-4}\int_{-\infty}^{+\infty}\frac{d^dp}{(2\pi)^d}\frac{1}{p^2+m_0^2}\frac{1}{(p+k)^2+m_0^2}+(\gamma-\gamma_0)+O(a^{d-2}).$$

Notice that the integral can be analytically computed in all dimensions, and the final result is

$$D(ka,m_0a,\gamma)\approx d_0\left(\frac{1}{4}k^2a^2+m_0^2a^2\right)^{\frac{d}{2}-2}{}_2F_1\left(2-\frac{d}{2},\frac{1}{2},\frac{3}{2},\frac{1}{\xi^2}\right)+\gamma-\gamma_0\equiv D_c,\tag{10}$$

where $d_0=\frac{1}{2}(1-\frac{d}{2})b_0$, ${}_2F_1$ is the Gauss hypergeometric function and $\xi=\sqrt{1+4m_0^2/k^2}$.

Let us in general denote by the label "c" the quantities occurring in the leading order in the homogeneous (continuum) expansion of D, and in particular $\gamma_c\equiv\gamma-\gamma_0$.

Following Ref. [8], we can exploit identities between hypergeometric functions to recast the above result into the form

$$D_c = a_0\,\xi^{d-3}(ka)^{d-4} + \gamma_c + \frac{b_0}{k^2a^2\xi^2}\,{}_2F_1\left(\frac{d-1}{2},1,\frac{d}{2},1-\frac{1}{\xi^2}\right)(m_0a)^{d-2}, \qquad (11)$$

where

$$a_0 \equiv \frac{1}{2}\frac{\Gamma(\frac{d}{2}-1)^2\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}\Gamma(d-2)}.$$

For $d=3$, Eq. (11) reduces to

$$D_c = \frac{1}{16\,ak} + \gamma_c - \frac{1}{8\pi ak}\,\operatorname{arccot}\frac{k}{2m_0}.$$

Eq. (11) is especially appropriate for the asymptotic expansion of D_c, because all the nonanalytic dependence on m_0a is explicitly factored out in the last term. In particular (after a rescaling $ka \to k$) we obtain the following behaviors:

$$A_{0c} = a_0(k^2)^{\frac{d}{2}-2} + \gamma_c, \qquad A_{nc} = \frac{1}{n!}\frac{4^n\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d-1}{2}-n)}a_0(k^2)^{\frac{d}{2}-2-n}.$$

Finally notice also that the zero-momentum value of D is related to the derivative of the gap equation with respect to the mass, and we can obtain the relationship

$$D_0 \equiv D(0,m_0a,\gamma) \approx d_0\,(m_0a)^{d-4} + \gamma_c - \frac{3d}{8}c\,b_0\,(m_0a)^{d-2} - \delta_0\,m_0^2a^2, \qquad (12)$$

and as a consequence

$$D_{0c} \equiv d_0(m_0a)^{d-4} + \gamma_c.$$

V. ASYMPTOTIC EXPANSION OF THE RENORMALIZED COUPLING

We now recall from the literature the expression of the $O(1/N)$ contribution to the (unrenormalized) self-energy of the fundamental quanta:

$$\Sigma_1(p,m_0) = \Sigma_{1a}(p,m_0) + \Sigma_{1b}(p,m_0),$$

where

$$\Sigma_{1a}(p,m_0) = \int_{-\pi}^{\pi}\frac{d^dk}{(2\pi)^d}\frac{\Delta(k,m_0,\gamma)}{\overline{(p+k)}^2+m_0^2},$$
$$\Sigma_{1b}(m_0) = \frac{1}{2}\Delta(0,m_0,\gamma)\int_{-\pi}^{\pi}\frac{d^dk}{(2\pi)^d}\Delta(k,m_0,\gamma)\frac{\partial}{\partial m_0^2}\Delta^{-1}(k,m_0,\gamma).$$

Again we might perform an asymptotic expansion of this expression, on the lines traced in Ref. [6], recovering in the scaling limit the (unrenormalized) continuum contribution, and in principle we might evaluate the first nonleading contribution.

However for our purposes it is much more convenient to work directly with quantities chosen in such a way that all renormalization effects are automatically removed, i.e., quantities whose scaling limit is a finite, renormalization-group invariant, amplitude.

The simplest such object is the so-called "renormalized coupling" g_r, whose continuum (scaling) value g_r^* has been computed to $O(1/N)$ in Ref. [4].

Actually the formal expression derived in Ref. [4] is correct also for the lattice versions of the model, when continuum propagators are replaced by their lattice counterparts. Parametrizing the result in terms of the renormalized mass m we therefore obtain

$$g_r(m,\gamma) = m^{d-4}\Delta(0,m,\gamma)\left[1+\frac{1}{N}g_r^{(1)}(m,\gamma)+O\left(\frac{1}{N^2}\right)\right], \tag{13}$$

where $g_r^{(1)}$ is given by

$$\begin{aligned} g_r^{(1)}(m,\gamma) = {} & \Delta(0.m,\gamma)\frac{\partial\Delta^{-1}(0,m,\gamma)}{\partial m^2}\left(\Sigma_{1a}(0,m)+\Sigma_{1b}(m)-m^2\frac{\partial\Sigma_{1a}}{\partial p^2}\bigg|_0\right) \\ & + \left(2\frac{\partial\Sigma_{1a}(0,m)}{\partial m^2}+\frac{\partial\Sigma_{1b}(m)}{\partial m^2}-2\frac{\partial\Sigma_{1a}}{\partial p^2}\bigg|_0-2\Delta^{-1}(0,m,\gamma)T(m)\right), \end{aligned} \tag{14}$$

and we defined

$$T(m_0) = \int_{-\pi}^{\pi}\frac{d^dk}{(2\pi)^d}\left(\frac{\Delta(k,m_0,\gamma)}{\bar{k}^2+m_0^2}\right)^2.$$

It is now a matter of trivial algebraic manipulations to show that the expression for $g_r^{(1)}$ can be cast into the form

$$\begin{aligned} g_r^{(1)} = {} & \frac{1}{D_0}\int_{-\pi}^{\pi}\frac{d^dk}{(2\pi)^d}\Bigg\{\frac{1}{D}\left(\frac{1}{2}\frac{\partial}{\partial m^2}\left[\frac{\partial D}{\partial m^2}+\frac{2D_0}{\bar{k}^2+m^2}\right]-\frac{D_0}{(\bar{k}^2+m^2)^2}-\left[2D_0+m^2\frac{\partial D_0}{\partial m^2}\right]C\right) \\ & -\frac{1}{2\,D^2}\left[\frac{\partial D}{\partial m^2}+\frac{2D_0}{\bar{k}^2+m^2}\right]^2\Bigg\}. \end{aligned} \tag{15}$$

In the asymptotic expansion of $g_r^{(1)}$ we may again identify a continuum and a lattice contribution.

Renormalization-group theory insures us about the expected properties of these contributions. In particular the leading continuum term should be finite and should therefore require no UV counterterms. Its evaluation will however allow us to identify the IR counterterms needed in order to regularize the lattice term.

The lattice contribution in turn should not affect the leading (scaling) order, since g_r^* is an invariant amplitude which should not depend on the detailed form of the Hamiltonian. In turn, trivial power counting in a shows us that the first correction to scaling is generated by the leading lattice contribution, which is $O((ma)^{4-d})$.

We can prove the following identities, corresponding to similar results of Ref. [4]:

$$\begin{aligned}\frac{\partial D_c}{\partial m^2} &= \frac{2}{k^2+4m^2}\left[(d-3)D_c - D_{0c} + (4-d)\gamma_c\right],\\ \frac{\partial D_{0c}}{\partial m^2} &= \left(\frac{d}{2}-2\right)\frac{D_{0c}-\gamma_c}{m^2},\\ C_c &= \frac{1}{(k^2+m^2)^2}\left(\frac{4}{d}\frac{k^2}{k^2+m^2}-1\right).\end{aligned}$$

As a consequence we obtain in leading order the following continuum contribution, depending on the single (dimensionless) variable $x \equiv \gamma_c(ma)^{4-d}$:

$$\begin{aligned} g^{(1)}_{r,\mathrm{con}}(x) \approx{}& \frac{1}{D_{0c}}\int_{-\infty}^{+\infty}\frac{d^dk}{(2\pi)^d}\\ &\times\Bigg(-2\left[\frac{d-3}{k^2+4m^2}+\frac{\gamma_c}{D_c}\frac{4-d}{k^2+4m^2}+\frac{D_{0c}}{D_c}\frac{3m^2}{(k^2+m^2)(k^2+4m^2)}\right]^2+2\frac{(d-3)(d-5)}{(k^2+4m^2)^2}\\ &+\frac{\gamma_c(4-d)}{D_c}\left[\frac{2(d-5)}{(k^2+4m^2)^2}+\frac{3}{2}\frac{1}{(k^2+4m^2)(k^2+m^2)}+\frac{1}{2}\left(1-\frac{4}{d}\right)\frac{1}{(k^2+m^2)^2}\right]\\ &+\frac{2\gamma_c}{D_c}\frac{(4-d)m^2}{d(k^2+m^2)^3}\\ &+\frac{D_{0c}}{D_c}\frac{m^2}{k^2+m^2}\left[\frac{6(d-5)}{(k^2+4m^2)^2}+\frac{3}{2}\frac{d-8}{(k^2+4m^2)(k^2+m^2)}+\frac{2}{(k^2+m^2)^2}\right]\Bigg), \end{aligned} \tag{16}$$

which can be shown to correspond exactly to the result presented in Ref. [4].

In particular the fixed-point value is obtained by setting $x=0$, corresponding to $\gamma_c = 0$, that is the condition for the removal of the leading nonscaling behavior in the large-N limit.

From the above result we may immediately read off the structure of counterterms, just by taking the power series expansion in powers of m^2, and in particular by replacing D_c with A_{0c} and D_{0c} with $d_0(ma)^{d-4}$. The resulting (singular) expression is

$$\frac{(ma)^{4-d}}{d_0}\int_{-\infty}^{+\infty}\frac{d^dk}{(2\pi)^d}\left(-2\left[\frac{d-3}{k^2}+\frac{\gamma_c}{A_{0c}}\frac{4-d}{k^2}\right]^2+2\frac{(d-3)(d-5)}{(k^2)^2}+2\frac{\gamma_c}{A_{0c}}\left(d-4-\frac{1}{d}\right)\frac{4-d}{(k^2)^2}\right).$$

Let us now compute the lattice contribution. By applying the already defined asymptotic expansions we obtain:

$$\frac{\partial D}{\partial m^2}+\frac{2D_0}{\bar{k}^2+m^2} \approx A_1 + 2\frac{\gamma_c}{\bar{k}^2} - b_0\left(\frac{d}{2}C_0+\frac{1}{(\bar{k}^2)^2}\right)m^{d-2} + 2\left(A_2 - \frac{\gamma_c}{(\bar{k}^2)^2} - \frac{\delta_0}{\bar{k}^2}\right)m^2 + O(m^d)$$

and

$$\frac{1}{2}\frac{\partial}{\partial m^2}\left[\frac{\partial D}{\partial m^2}+\frac{2D_0}{\bar{k}^2+m^2}\right]-\frac{D_0}{(\bar{k}^2+m^2)^2}-\left[2\,D_0+m^2\frac{\partial D_0}{\partial m^2}\right]C$$
$$\approx A_2-\frac{\delta_0}{\bar{k}^2}-2\gamma_c\left(C_0+\frac{1}{(\bar{k}^2)^2}\right)+O(m^{d-2}).$$

Notice that the most singular contributions, $O(m^{d-4})$, have been removed in the above combinations.

Let us now find out the leading IR singularities of the terms appearing in the integral representing $g^{(1)}_{r,\mathrm{lat}}$. We must only notice that, for all $A_n(k)$, the singular behavior when $k\to 0$ is determined by the corresponding behavior of $A_{nc}(k)$, with corrections whose singularity is depressed by a factor k^2. As a consequence we obtain in the IR limit

$$\frac{1}{A_0}\left(A_1+2\frac{\gamma_c}{k^2}\right)\to\frac{2}{k^2}\left[d-3+(4-d)\frac{\gamma_c}{A_{0c}}\right], \tag{17}$$

$$\frac{1}{A_0}\left(A_2-\frac{\delta_0}{\bar{k}^2}-2\gamma_c(C_0+\frac{1}{(\bar{k}^2)^2})\right)\to\frac{2}{(k^2)^2}\left[(d-3)(d-5)+\left(d-4-\frac{1}{d}\right)(4-d)\frac{\gamma_c}{A_{0c}}\right].$$

These singularities are perfectly matched by the terms coming from the $T^{(\mathrm{IR})}$ expansion of the continuum contribution, as expected. As a consequence we are able to write down an exact, finite representation of the leading lattice contribution to $g_r^{(1)}$, taking the form

$$g^{(1)}_{r,\mathrm{lat}}(m,\gamma_c)\equiv\frac{(ma)^{4-d}}{d_0}\delta g^{(1)}(\gamma_c), \tag{18}$$

where

$$\delta g^{(1)}(\gamma_c)\equiv\int_{-\pi}^{\pi}\frac{d^dk}{(2\pi)^d}\left(\frac{1}{A_0}\left(A_2-\frac{\delta_0}{\bar{k}^2}-2\gamma_c\left(C_0+\frac{1}{(\bar{k}^2)^2}\right)\right)-\frac{1}{2}\left[\frac{1}{A_0}\left(A_1+2\frac{\gamma_c}{\bar{k}^2}\right)\right]^2\right)$$
$$-\int_{-\infty}^{+\infty}\frac{d^dk}{(2\pi)^d}\frac{2}{(k^2)^2}\left(\left[(d-3)(d-5)+\left(d-4-\frac{1}{d}\right)(4-d)\frac{\gamma_c}{A_{0c}}\right]\right.$$
$$\left.-\left[d-3+(4-d)\frac{\gamma_c}{A_{0c}}\right]^2\right). \tag{19}$$

It is worth observing that $g^{(1)}_{r,\mathrm{con}}(x)$ showed a nonanalyticity whose leading dependence was proportional to $x\ln x$. Correspondingly $\delta g^{(1)}(\gamma_c)$ has a nonanalytic $\gamma_c\ln\gamma_c$ dependence. The coefficients of these singularities match properly in order to reproduce an overall dependence proportional to $\gamma_c(ma)^{4-d}\ln ma$, as expected from general renormalization group arguments because of the anomalous dimension of the leading irrelevant operator.

VI. INTEGRAL REPRESENTATION FOR THE GAP EQUATION

For the purpose of actual numerical calculations one must find an efficient way of performing lattice momentum integrals. In practice this may be obtained by resorting to parametric (Feynman and Schwinger) representations of the lattice propagators. These representations are especially useful in the case corresponding to the standard nearest-neighbor Hamiltonian.

Let us first consider the integral appearing in the gap equation

$$\beta + 2\gamma m_0^2 = \int_{-\pi}^{\pi} \frac{d^d p}{(2\pi)^d} \frac{1}{\hat{p}^2 + m_0^2} \equiv \Xi(m_0^2).$$

Introducing Schwinger's proper time representation we obtain

$$\Xi(m_0^2) = \int_0^\infty d\alpha\, e^{-\alpha m_0^2} \int_{-\pi}^{\pi} \frac{d^d p}{(2\pi)^d} e^{-2\alpha \sum_\mu (1-\cos p_\mu)} = \int_0^\infty d\alpha\, e^{-\alpha m_0^2} \left[e^{-2\alpha} I_0(2\alpha) \right]^d, \tag{20}$$

where I_0 is the standard modified Bessel function, admitting for large values of its argument the following asymptotic expansion:

$$e^{-2\alpha} I_0(2\alpha) \approx \frac{1}{(4\pi\alpha)^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n+\frac{1}{2})}{n!\,\Gamma(\frac{1}{2}-n)} \frac{1}{(4\alpha)^n}.$$

By a proper change of variables the numerical evaluation of the above integral is now possible even in the small m_0^2 regime.

When $d = 3$ it is also possible to obtain an analytical expression for $\Xi(0)$ [5], which was first derived in Ref. [9]:

$$\beta_c = \Xi(0) = \frac{\varkappa^2 + 1}{\pi^2} K^2(\varkappa), \tag{21}$$

where K is the complete elliptic integral of the first kind and $\varkappa = (2-\sqrt{3})(\sqrt{3}-\sqrt{2})$, and the resulting numerical value is $\beta_c = 0.252731009858663...$

We can compute $\Xi(m_0^2)$ as an asymptotic expansion around $m_0^2 = 0$:

$$\Xi(m_0^2) = \sum_{n=0}^{\infty} a_n m_0^{2n} + b_n m_0^{2n+d-2}. \tag{22}$$

It is conceivable that when $d = 3$ the coefficients a_n of the expansion can be computed analytically in terms of elliptic integrals, but we contented ourselves with a numerical calculation of the "lattice terms" appearing in the analytic part of the expansion, while it is always possible to obtain closed form expressions for the coefficients b_n of the nonanalytic part.

We can obtain explicit expressions for the coefficients in the expansion (22) by subtracting a proper number of terms of the asymptotic expansion of the Bessel function raised to the power d. Let us label the coefficients of this expansion according to the equation

$$\left[e^{-2\alpha} I_0(2\alpha)\right]^d \approx \frac{1}{(4\pi\,\alpha)^{\frac{d}{2}}} \sum_{n=0}^{\infty} c_n(d)\,\alpha^{-n}; \tag{23}$$

the coefficients can be computed recursively from the equations

$$\begin{aligned} c_0(d) &= 1, \\ c_n(d) &= \frac{1}{n} \sum_{k=1}^{n} (kd - n + k) \frac{(-1)^k\,\Gamma(\frac{1}{2}+k)}{4^k k!\,\Gamma(\frac{1}{2}-k)}\, c_{n-k}. \end{aligned} \tag{24}$$

Let us add to the integrand in Eq. (20) the formally vanishing term

$$-\frac{1}{(4\pi\,\alpha)^{\frac{d}{2}}} \sum_{n=0}^{\infty} \frac{(-\alpha\, m_0^2)^n}{n!} \sum_{m=0}^{n-1} c_m(d)\,\alpha^{-m} + \frac{1}{(4\pi\,\alpha)^{\frac{d}{2}}} \sum_{n=0}^{\infty} c_n(d)\,\alpha^{-n} \sum_{m=n+1}^{\infty} \frac{(-\alpha\, m_0^2)^m}{m!},$$

where we have only interchanged the order of the summations in the two contributions. It is now possible to group the first contribution with the original integrand and recognize that the resulting combination defines an analytic function of m_0^2, since each coefficient

$$a_n \equiv \int_0^\infty d\alpha\, \frac{(-\alpha)^n}{n!} \left\{ \left[e^{-2\alpha} I_0(2\alpha)\right]^d - \frac{1}{(4\pi\,\alpha)^{\frac{d}{2}}} \sum_{m=0}^{n-1} c_m(d)\,\alpha^{-m} \right\} \tag{25}$$

in the corresponding power series expansion is a finite, UV and IR regulated integral for any d in the range $2 < d < 4$. In turn, the integration of the second contribution may be represented, after trivial resummations and rescalings, in the form

$$\frac{1}{(4\pi)^{\frac{d}{2}}} \sum_{n=0}^{\infty} c_n(d)\, k_n(d)\, (m_0^2)^{n+\frac{d}{2}-1},$$

where

$$k_n(d) \equiv \int_0^\infty dx\, x^{-n-\frac{d}{2}} \left[e^{-x} - \sum_{m=0}^{n} \frac{(-x)^m}{m!} \right]$$

are finite UV and IR regulated integrals in the range $2 < d < 4$; integrating by parts we obtain a recursive equation for k_n, which can be solved immediately, obtaining $k_n = \Gamma(1 - \frac{d}{2} - n)$, and consequently

$$b_n = \frac{1}{(4\pi)^{\frac{d}{2}}}\, c_n(d)\, \Gamma\!\left(1 - \frac{d}{2} - n\right). \tag{26}$$

It is now trivial to set $d = 3$ in Eq. (24) to obtain an explicit recursive expression for b_n.

To compute numerical values of a_n in $d = 3$, it is not practical to use Eq. (25), given the slow convergence of the integration for large α. We found it more convenient to split the α integration in Eq. (20) at $\alpha = 1$. For $\alpha \le 1$ we can expand in powers of m_0^2 under the integration sign and integrate term by term. For $\alpha \ge 1$, we subtract $n + l$ terms of the expansion (23), expand the integrand in powers of m_0^2 up to $O(m_0^{2n})$, and integrate term by term; while the resulting integrals converge for any $l \ge 0$, it is useful to set $l \ge 3$ to ensure fast convergence of the integration for large α. The integral of the regulator can be computed analytically in terms of the incomplete Gamma function, and the singular part of its asymptotic expansion reproduces the singular part of the expansion (22).

Collecting all numerical and analytical results, we obtain for $d = 3$ the following asymptotic expansion:

$$
\begin{aligned}
\Xi(m_0^2) = \beta_c &- \frac{1}{4\pi} m_0 - 0.012164158583022\, m_0^2 + \frac{1}{32\,\pi} m_0^3 + 0.00083776240606293\, m_0^4 \\
&- \frac{11}{2560\,\pi} m_0^5 - 0.000066743211781194\, m_0^6 + \frac{281}{430080\,\pi} m_0^7 + 5.7884488124445 \times 10^{-6} m_0^8 \\
&- \frac{71}{655360\,\pi} m_0^9 - 5.320777180475 \times 10^{-7} m_0^{10} + \frac{7783}{403701760\,\pi} m_0^{11} \\
&+ 5.107544566044 \times 10^{-8} m_0^{12} - \frac{70289}{19377684480\,\pi} m_0^{13} + O(m_0^{14}),
\end{aligned} \tag{27}
$$

which gives an error smaller than 10^{-14} in the range $0 \le m_0^2 \le 0.1$.

It is immediate to extract from this expansion the 3-d large-N nearest-neighbor model values

$$
\gamma_0 \approx -0.0060820792915113, \qquad \delta_0 \approx 0.00083776240606293.
$$

The asymptotic expansion of $\Xi^{(n)}(m_0^2)$, the nth derivative of $\Xi(m_0^2)$ with respect to m_0^2, is easily obtained from the above expression. There is an obvious precision loss, but in the above-mentioned range the error in the second derivative is still smaller than 10^{-10}.

In the following, we will need to compute $\Xi^{(n)}$ in a fast and accurate way, for generic values of m_0^2; to this purpose, we tabulated Ξ for values of m_0^2 on a uniform grid with step $h = 10^{-3}$ and compute $\Xi^{(n)}(m_0^2)$ by n+4-point Lagrange interpolation.

VII. INTEGRAL REPRESENTATIONS FOR THE EFFECTIVE PROPAGATOR

Let us now recall from Ref. [6] the following basic result:

$$D(k, m_0, 0) = \frac{1}{2} \int_0^1 dx \int_{-\pi}^{\pi} \frac{d^d q}{(2\pi)^d} \frac{1}{[m_0^2 + 2\sum_\mu (1 - z_\mu \cos q_\mu)]^2},$$

where $z_\mu = \sqrt{1 - x(1-x)\hat{k}_\mu^2}$.

Using Schwinger representation we then obtain:

$$\begin{aligned} D(k, m_0, 0) &= \frac{1}{2} \int_0^1 dx \int_0^\infty d\alpha\, \alpha\, e^{-\alpha m_0^2} \int_{-\pi}^{\pi} \frac{d^d q}{(2\pi)^d} e^{-2\alpha \sum_\mu (1 - z_\mu \cos q_\mu)} \\ &= \frac{1}{2} \int_0^1 dx \int_0^\infty d\alpha\, \alpha\, e^{-\alpha m_0^2} \prod_\mu \Big[e^{-2\alpha} I_0(2\alpha z_\mu) \Big]. \end{aligned} \qquad (28)$$

Alternatively we might use the Schwinger representation directly and obtain

$$D(k, m_0, 0) = \frac{1}{2} \int_0^\infty ds \int_0^\infty dt\, e^{-(s+t)m_0^2} \prod_\mu \left[e^{-2(s+t)} I_0(2\sqrt{s^2 + t^2 + 2s\,t \cos k_\mu}) \right],$$

that can be reduced to the previous one by the variable change $s = x\,\alpha$, $t = (1-x)\,\alpha$.

The direct numerical evaluation of Eq. (28), and especially of its derivatives with respect to m_0^2, in $d = 3$ is difficult, particularly for small values of k or m_0; the convergence can be improved dramatically by adding and subtracting a symmetric combination of Bessel functions:

$$\begin{aligned} D(k, m_0, 0) &= \frac{1}{2} \int_0^1 dx \int_0^\infty d\alpha\, \alpha\, e^{-\alpha(6 + m_0^2)} \Big[\prod_\mu I_0(2\alpha z_\mu) - I_0^3(2\alpha \bar{z}) \Big] \\ &- \frac{1}{2} \int_0^1 dx\, \frac{1}{\bar{z}^2}\, \Xi' \left(\frac{m_0^2 + 6}{\bar{z}} - 1 \right), \end{aligned} \qquad (29)$$

where $\bar{z} = \frac{1}{3}(z_1 + z_2 + z_3)$; the subtracted integral and its first few derivatives with respect to m_0^2 can now be computed accurately by Gauss-Legendre integration on a small grid; the integration in x of Ξ' and of its derivatives is also easy, once a few singular terms of Eq. (22) have been subtracted.

In order to identify the lattice contributions to the asymptotic expansion of D we must expand the integrand in powers of m_0^2. The IR singularities turn into unsuppressed positive powers of α, present in the large-α regime when $z_\mu \to 1$. These singularities become worse and worse with increasing powers of m_0^2. In order to classify them according to their degree

we need to consider the homogeneous expansion of the integrand in powers of α^{-1} and of $\sqrt{1-z_\mu^2} \equiv x(1-x)\hat{k}_\mu^2$.

Recalling once more the asymptotic expansion of the Bessel function, we can write down the homogeneous expansion in the form:

$$\prod_\mu e^{-2\alpha} I_0(2\alpha z_\mu) \approx \frac{e^{-\alpha x(1-x)\hat{k}^2}}{(4\pi\alpha)^{\frac{d}{2}}} \sum_{n=0}^{\infty} C_n[\, d, \alpha x(1-x)\hat{k}_\mu^2]\, \alpha^{-n}, \tag{30}$$

where in turn

$$C_n[\, d, \alpha x(1-x)\hat{k}_\mu^2] \equiv \sum_{m=0}^{2n} \gamma_{nm}(d)[\alpha x(1-x)\hat{k}^2]^m \tag{31}$$

and $\gamma_{nm}(d)$ may show a dependence on the ratios $\hat{k}^{2p}/(\hat{k}^2)^p$.

Repeating the procedure developed in the previous section we may now obtain the following decomposition:

$$D(k, m_0, 0) = \sum_{n=0}^{\infty} R_n(k) m_0^{2n} + D_s(k, m_0), \tag{32}$$

where

$$R_n(k) \equiv \frac{(-1)^n}{2\, n!} \int_0^1 dx \int_0^\infty d\alpha\, \alpha^{n+1} \left[\prod_\mu e^{-2\alpha} I_0(2\alpha z_\mu) - \frac{e^{-\alpha x(1-x)\hat{k}^2}}{(4\pi\alpha)^{\frac{d}{2}}} \sum_{m=0}^{n} C_m\, \alpha^{-m} \right] \tag{33}$$

is the part of $A_n(k) \equiv R_n(k) + S_n(k)$ which shows a regular dependence on k in the $k \to 0$ limit, and

$$D_s(k, m_0) \equiv \frac{1}{2} \int_0^1 dx \int_0^\infty d\alpha \frac{e^{-\alpha x(1-x)\hat{k}^2}}{(4\pi\alpha)^{\frac{d}{2}}} \sum_{n=0}^{\infty} \left[e^{-\alpha m_0^2} - \sum_{m=0}^{n-1} \frac{(-\alpha m_0^2)^m}{m!} \right] C_n\, \alpha^{1-n} \tag{34}$$

is a calculable expression including all the singular dependence on k and admitting an asymptotic expansion in the form

$$D_s(k, m_0) \approx \sum_{n=0}^{\infty} [S_n(k) m_0^{2n} + B_n(k) m_0^{2n+d-2}].$$

In order to compute $D_s(k, m_0)$ let us notice that it may be also expressed in the form

$$D_s(k, m_0) = \sum_{n=0}^{\infty} \sum_{m=0}^{2n} \gamma_{nm}(d) I_{nm}(m_0^2) \tag{35}$$

where

$$I_{nm}(m_0^2) \equiv \frac{1}{2} \int_0^1 dx [x(1-x)\hat{k}^2]^m \int_0^\infty d\alpha \frac{e^{-\alpha x(1-x)\hat{k}^2}}{(4\pi\alpha)^{\frac{d}{2}}} \left[e^{-\alpha m_0^2} - \sum_{m=0}^{n-1} \frac{(-\alpha m_0^2)^m}{m!} \right] \alpha^{1+m-n} \tag{36}$$

enjoys the property

$$\frac{dI_{nm}}{dm_0^2} = -I_{n-1,m}.$$

Therefore we only need to evaluate

$$\begin{aligned}
I_{0m}(m_0^2) &\equiv \frac{1}{2}\int_0^1 dx[x(1-x)\hat{k}^2]^m \int_0^\infty d\alpha\, e^{-\alpha x(1-x)\hat{k}^2-\alpha m_0^2}\,\frac{\alpha^{1-\frac{d}{2}+m}}{(4\pi)^{\frac{d}{2}}} \\
&= \frac{1}{2}\frac{\Gamma(2-\frac{d}{2}+m)}{(4\pi)^{\frac{d}{2}}}\int_0^1 dx\frac{[x(1-x)\hat{k}^2]^m}{[m_0^2+x(1-x)\hat{k}^2]^{2-\frac{d}{2}+m}} \\
&= \frac{1}{2}\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}(-\hat{k}^2)^m\left(\frac{\partial}{\partial\hat{k}^2}\right)^m\int_0^1\frac{dx}{[m_0^2+x(1-x)\hat{k}^2]^{2-\frac{d}{2}}}.
\end{aligned} \tag{37}$$

A trivial comparison with previous results shows that the last integral is directly related to the continuum propagator by the replacement $k^2 \to \hat{k}^2$, and therefore, setting $\hat{\xi}^2 = 1 + 4m_0^2/\hat{k}^2$ we obtain

$$I_{0m}(m_0^2) = (-\hat{k}^2)^m\left(\frac{\partial}{\partial\hat{k}^2}\right)^m\left[a_0\hat{\xi}^{d-3}(\hat{k}^2)^{\frac{d}{2}-2} + \frac{b_0}{\hat{k}^2\hat{\xi}^2}\,{}_2F_1\!\left(\frac{d-1}{2},1,\frac{d}{2},1-\frac{1}{\hat{\xi}^2}\right)m_0^{d-2}\right], \tag{38}$$

where we must appreciate that the above expression is naturally decomposed into an analytic and a nonanalytic term, which implies that we can immediately relate all coefficients $S_n(k)$ to the derivatives with respect to $\hat{k}^2$ of $\hat{\xi}^{d-3}(\hat{k}^2)^{\frac{d}{2}-2}$.

Straightforward manipulations lead to the general relationship

$$S_n(k) = \frac{a_0}{n!}(\hat{k}^2)^{\frac{d}{2}-2}\left(\frac{4}{\hat{k}^2}\right)^n\sum_{p=0}^{n}\frac{(-1)^p\,\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d-1}{2}-n+p)}\left(\frac{\hat{k}^2}{4}\right)^p\sum_{q=0}^{2p}\frac{(-1)^q\,\Gamma(\frac{d}{2}-1-n+p)}{\Gamma(\frac{d}{2}-1-n+p-q)}\,\gamma_{pq}(d). \tag{39}$$

The above expression brings into evidence a peculiar feature exhibited by the functions $S_n(k)$ when $d = 3$. In this case the arguments of the function $\Gamma(1-n+p)$ appearing in the denominator are integer nonpositive numbers whenever $p \neq n$, and therefore the corresponding contributions to the sum vanish, giving

$$S_n(k) = (-1)^n\frac{a_0}{n!}(\hat{k}^2)^{-\frac{1}{2}}\sum_{q=0}^{2n}\frac{(-1)^q\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-q)}\gamma_{nq}(3). \tag{40}$$

When $d = 3$ the singular contributions to $A_n(k)$ are proportional to $(\hat{k}^2)^{-\frac{1}{2}}$ for all n, and this is basically a consequence of the vanishing dependence on ξ in the analytic part of the continuum propagator.

The simplest example of this procedure is offered by the integral representation of $A_0(k,0)$:

$$A_0(k,0) = a_0\,(\hat{k}^2)^{\frac{d}{2}-2} + R_0(k), \tag{41}$$

where

$$R_0(k) \equiv \frac{1}{2}\int_0^1 dx \int_0^\infty d\alpha\, \alpha \left[\prod_\mu e^{-2\alpha} I_0(2\alpha z_\mu) - \frac{e^{-\alpha x(1-x)\hat{k}^2}}{(4\pi\alpha)^{\frac{d}{2}}}\right]. \tag{42}$$

This representation is worth a few observations. Notice that the explicit term might be simply obtained from the expression of A_{0c} by the replacement $k^2 \to \hat{k}^2$, and therefore it reproduces exactly the singular behavior of A_0 when $k \to 0$. Moreover $R_0(k)$ has, by construction, a finite limit when $k \to 0$. It is easy to check that $R_0(0) = -\gamma_0$, thus verifying explicitly that the choice $\gamma = \gamma_0$ leads to the $O(a^{4-d})$ improvement of the lattice massless propagator.

By the same technique we obtain

$$A_1(k) = a_0\left[\frac{2(d-3)}{\hat{k}^2} + \frac{(6-d)(4-d)}{16}\frac{\hat{k}^4}{(\hat{k}^2)^2} - \frac{(8-d)}{16}\right](\hat{k}^2)^{\frac{d}{2}-2} + R_1(k), \tag{43}$$

where

$$R_1(k) \equiv -\frac{1}{2}\int_0^1 dx \int_0^\infty d\alpha\, \alpha^2 \left[\prod_\mu e^{-2\alpha} I_0(2\alpha z_\mu) - \frac{e^{-\alpha x(1-x)\hat{k}^2}}{(4\pi\alpha)^{\frac{d}{2}}}\left(1 + \frac{d}{16\alpha} + \frac{1}{4}x(1-x)\hat{k}^2 - \frac{1}{4}\alpha x^2(1-x)^2\hat{k}^4\right)\right]. \tag{44}$$

Again we notice that the explicit term reproduces the leading singularities, and moreover $R_1(0) = -\delta_0$.

We also mention the result for $A_2(k)$, since it will be needed in the evaluation of $\delta g^{(1)}$:

$$A_2(k) = \frac{1}{2}a_0\left[\frac{a_{2,0}}{(\hat{k}^2)^2} + \frac{a_{2,1}}{\hat{k}^2} + a_{2,2}\right](\hat{k}^2)^{\frac{d}{2}-2} + R_2(k), \tag{45}$$

where

$$\begin{aligned} a_{2,0} &= 4(d-3)(d-5), \\ a_{2,1} &= \frac{(d-3)(d-12)}{8} + \frac{(d-3)(d-6)(d-8)}{8}\frac{\hat{k}^4}{(\hat{k}^2)^2}, \\ a_{2,2} &= \frac{(d-16)(d-8)}{512} + \frac{(d-4)(d-6)(d-8)}{256}\frac{\hat{k}^4}{(\hat{k}^2)^2} \\ &\quad + \frac{(d-4)(d-6)(d-8)}{64}\frac{\hat{k}^6}{(\hat{k}^2)^3} + \frac{(d-4)(d-6)(d-8)(d-10)}{512}\frac{(\hat{k}^4)^2}{(\hat{k}^2)^4}, \end{aligned}$$

and we obtain

$$R_2(k) \equiv \frac{1}{4}\int_0^1 dx \int_0^\infty d\alpha\, \alpha^3 \left[\prod_\mu e^{-2\alpha} I_0(2\alpha z_\mu) - \frac{e^{-\alpha x(1-x)\hat{k}^2}}{(4\pi\alpha)^{\frac{d}{2}}}(1 + r_{2,1}(k) + r_{2,2}(k))\right], \tag{46}$$

where in turn

$$
\begin{aligned}
r_{2,1}(k) &= \frac{d}{16\,\alpha} + \frac{1}{4}x(1-x)\hat{k}^2 - \frac{1}{4}\alpha x^2(1-x)^2\hat{k}^4, \\
r_{2,2}(k) &= \frac{d(d+8)}{512\,\alpha^2} + \frac{1}{64\,\alpha}(d+2)x(1-x)\hat{k}^2 + \frac{1}{64}(2(\hat{k}^2)^2 + (8-d)\hat{k}^4)x^2(1-x)^2 \\
&\quad - \frac{1}{16}\alpha x^3(1-x)^3(2\hat{k}^6 + \hat{k}^4\hat{k}^2) + \frac{1}{32}\alpha^2 x^4(1-x)^4(\hat{k}^4)^2.
\end{aligned}
$$

An obvious relationship exists between the values $R_n(0)$ and the coefficients of the analytic part of the asymptotic expansion of the function $\Xi'(m_0^2)$. One easily finds that $R_n(0) = \frac{1}{2}(n+1)a_{n+1}$.

It should be by now clear that a trivial generalization of the same technique allows for an explicit, albeit more cumbersome, evaluation of all the functions $B_n(k)$. We could verify that the correct expressions for $B_0(k)$ and $B_1(k)$ are reproduced, and we computed $B_2(k)$ for a better accuracy check of our numerical estimates. The result is too cumbersome to be reported here.

In practice, A_0, A_1, and A_2 can be computed numerically in $d = 3$ exploiting the subtraction (29); since

$$
e^{-6t}\left[\prod_\mu I_0(2\alpha z_\mu) - I_0^3(2\alpha\bar{z})\right] \approx e^{-\alpha x(1-x)\hat{k}^2}\, x^2(1-x)^2\, \frac{3\hat{k}^4 - (\hat{k}^2)^2}{48(4\pi\alpha)^{3/2}}, \tag{47}
$$

the subtraction is enough to regularize A_1 and A_2. Let us consider, e.g., A_1; by straightforward manipulations we obtain

$$
\begin{aligned}
A_1(k) = -\frac{1}{2}\int_0^1 dx\bigg\{\int_0^1 d\alpha\,\alpha^2\, e^{-6\alpha}\left[\prod_\mu I_0(2\alpha z_\mu) - I_0^3(2\alpha\bar{z})\right] \\
+ \frac{1}{\bar{z}^3}\,\Xi''\left(\frac{6}{\bar{z}} - 1\right) - \frac{1}{16\pi(x(1-x)\hat{k}^2)^{3/2}}\bigg\};
\end{aligned} \tag{48}
$$

it is useful to overregulate the x integration by subtracting the r.h.s. of Eq. (47), integrated over α. For A_2 we follow the same procedure; the subtractions are more complicated and not worth writing here. The computation of A_0 is similar but of course easier.

We verified explicitly that A_n, computed in this way, and B_n are consistent with Eq. (7) for $n \le 2$.

VIII. NUMERICAL RESULTS

It is worthwhile to present a selection of the numerical results that we obtained in $d = 3$ for the nearest-neighbor formulation ($c = -\frac{1}{12}$).

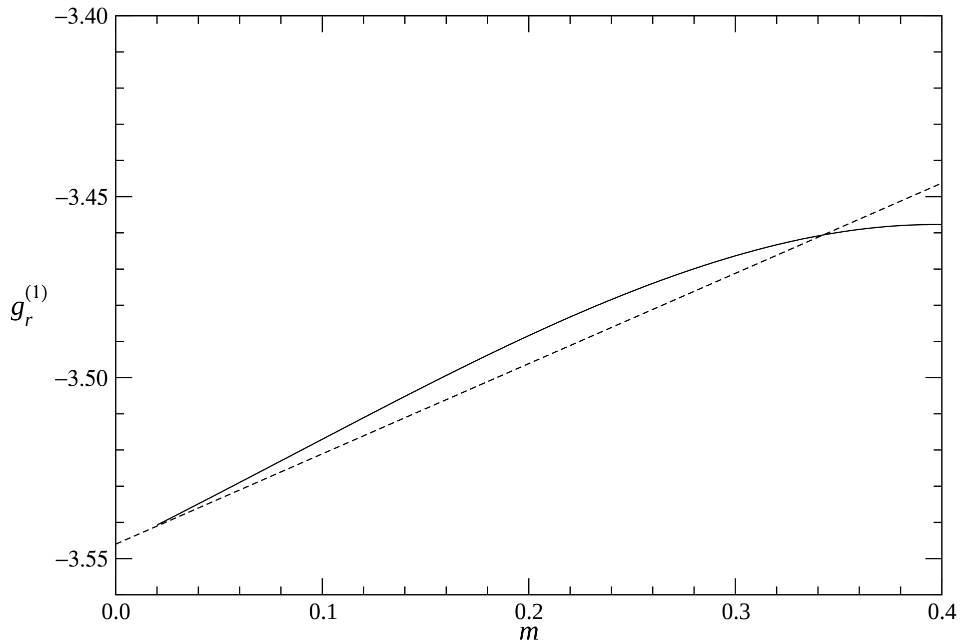

FIG. 3: $g_r^{(1)}$ (solid line) and $g_{r,\mathrm{con}}^{(1)} + g_{r,\mathrm{lat}}^{(1)}$ (dashed line) vs. m for $\gamma_c = 0$.

In Figs. 3, 4, and 5 we compare $g_r^{(1)}(m, \gamma_c)$, obtained by direct evaluation of Eq. (15), with $g_{r,\mathrm{con}}^{(1)}(\gamma_c (ma)^{4-d}) + g_{r,\mathrm{lat}}^{(1)}(m, \gamma_c)$, obtained from Eqs. (16), (18), and (19), for three values of γ_c; of special interest is the value $\gamma_c = -\gamma_0$, i.e., $\gamma = 0$, corresponding to the nonlinear σ model.

$\delta g^{(1)}(\gamma_c)$ is plotted in Fig. 6; of special interest is the value

$$\delta g^{(1)}(0) = 0.0049699...$$

IX. CONSTRUCTION OF THE IMPROVED HAMILTONIAN

As we mentioned in the Introduction, the improvement procedure aims at a systematic cancellation of the next-to-leading effects in the invariant amplitudes. The renormalization-group theory insures us that a single choice of γ exists such that this cancellation occurs in all amplitudes. It is therefore sufficient to find the value $\gamma = \gamma^*$ for which the cancellation occurs in the renormalized coupling.

In the context of the $1/N$ expansion we may assume γ^* to admit an expansion in powers of $1/N$:

$$\gamma^* = \gamma_0^* + \frac{1}{N}\gamma_1^* + O\left(\frac{1}{N^2}\right).$$

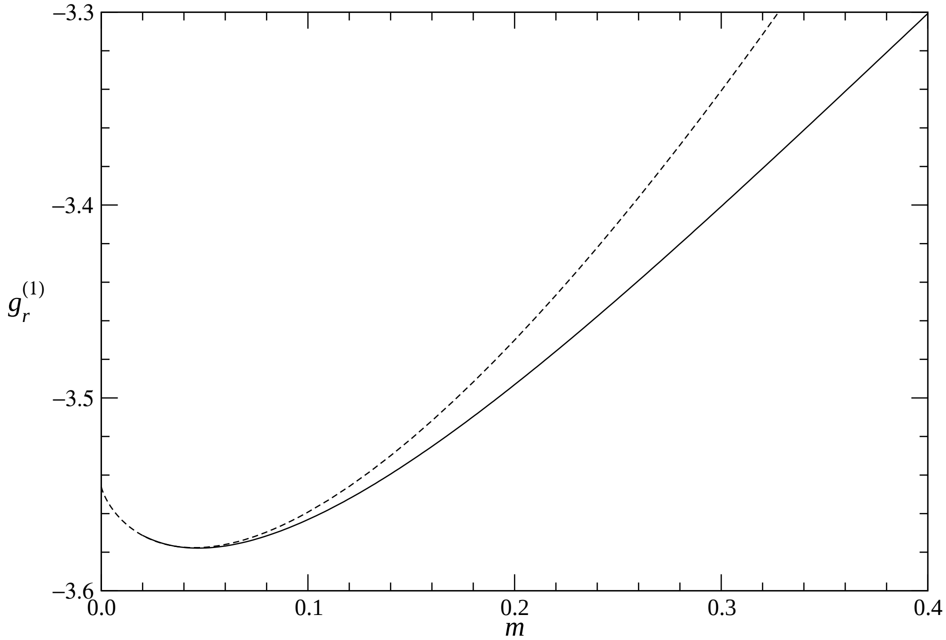

FIG. 4: $g_r^{(1)}$ (solid line) and $g_{r,\mathrm{con}}^{(1)} + g_{r,\mathrm{lat}}^{(1)}$ (dashed line) vs. m for $\gamma_c = -\gamma_0$, i.e., for the nonlinear σ model.

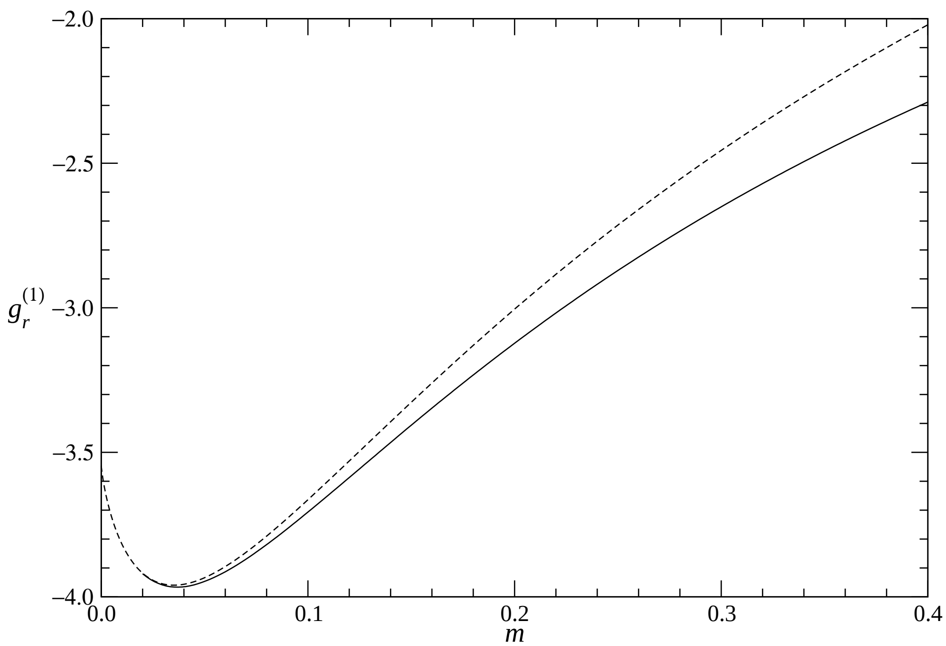

FIG. 5: $g_r^{(1)}$ (solid line) and $g_{r,\mathrm{con}}^{(1)} + g_{r,\mathrm{lat}}^{(1)}$ (dashed line) vs. m for $\gamma_c = 0.1$.

We have already recognized in Sec. III that $\gamma_0^* = \gamma_0$. We may therefore define

$$\gamma_c^* \equiv \gamma^* - \gamma_0 \approx \frac{1}{N}\gamma_1^*.$$

Substituting this result in the expression of g_r and expanding in powers of m we then

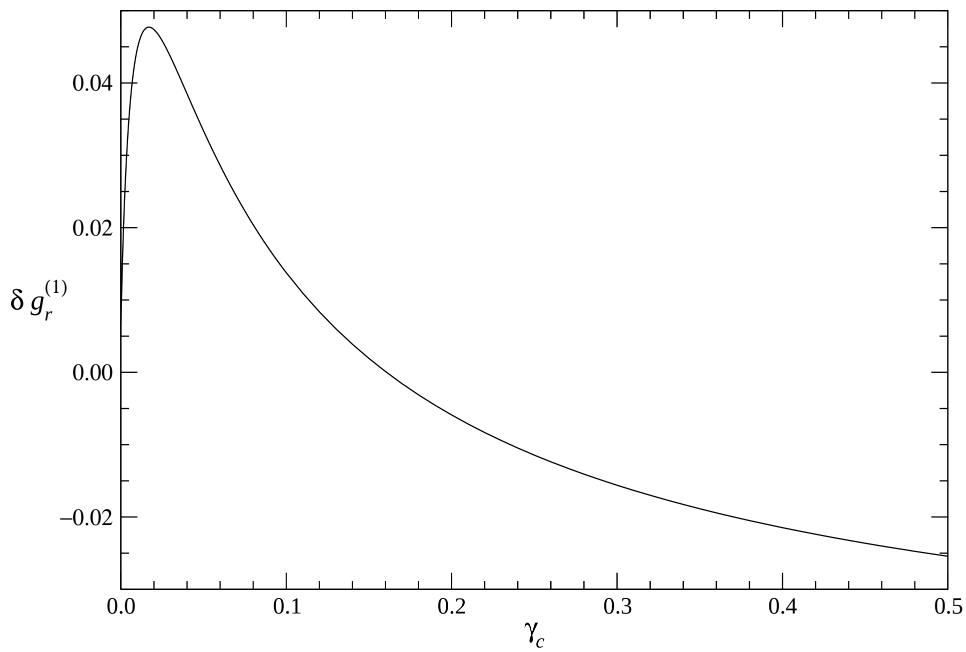

FIG. 6: $\delta g^{(1)}$ vs. γ_c.

obtain:

$$g_r(ma, \gamma^*) \approx \frac{1}{d_0}\left[1 + \frac{1}{N} g^{(1)}_{r,\mathrm{con}}(0) + \frac{1}{N}\frac{1}{d_0}(\delta g_1(0) - \gamma_1^*)(ma)^{4-d} + O(m^2a^2) + O\left(\frac{1}{N^2}\right)\right].$$

We then recognize that the condition for the cancellation of the first nonleading contribution is

$$\gamma_1^* = \delta g_1(0) \equiv \int_{-\pi}^{\pi} \frac{d^dk}{(2\pi)^d}\left(\frac{1}{\tilde{A}_0}\left(A_2 - \frac{\delta_0}{\bar{k}^2}\right) - \frac{1}{2}\left(\frac{A_1}{\tilde{A}_0}\right)^2\right) + \int_{-\infty}^{+\infty} \frac{d^dk}{(2\pi)^d}\frac{4(d-3)}{(k^2)^2},$$

where $\tilde{A}_0 \equiv A_0(k, \gamma_0)$.

X. CONCLUSIONS

In the case of the three-dimensional $O(N)$ models with standard nearest-neighbor interactions, our analytical results led us to the prediction

$$\gamma^* \cong -0.00608207 + \frac{1}{N}\, 0.0049699 + O\left(\frac{1}{N^2}\right). \tag{49}$$

We might now obtain an estimate for the value N_c for which γ^* vanishes. The numerical value of the present estimate is admittedly not very promising, but, as we mentioned in the introduction, the essential feature of γ_1^* is its positive sign, suggesting that N_c may exist and possibly be within the range of convergence of the $1/N$ expansion.

The physical interpretation of N_c amounts to the statement that, for this special value of N, the lattice version of the nonlinear σ model ($\gamma = 0$) in three dimensions shows the absence of leading irrelevant operators in the expansion of the Hamiltonian into scaling fields.

For all values $N \leq N_c$, one would get $\gamma^* > 0$, and as a consequence one may proceed to analyze the models in their improved version, both by numerical Monte Carlo methods and by perturbative expansion techniques. Numerical evidence shows that this is actually the case for the physically interesting cases $N \leq 3$. It might be interesting to perform a numerical study with the purpose of estimating N_c. A naïve extrapolation from the known numerical values of $\gamma^*(N)$,

$$\gamma^*(1) = 0.0159(3), \quad \gamma^*(2) = 0.0078(2), \quad \gamma^*(3) = 0.0043(4), \quad \gamma^*(4) = 0.0021(7),$$

obtained from Refs. [1], [10], [11], and [12] respectively using the formula $\gamma = \beta_c^2/(8\lambda N)$, suggests $N_c \simeq 5 \div 6$.

Concerning the extension of the $1/N$ expansion itself to the region $N < N_c$, we must cautiously mention that some dramatic change in the analytical behavior of the function $\gamma^*(N)$ may certainly occur at $N = N_c$. It is not possible to perform a strong-coupling expansion of the models when $\gamma < 0$, as one may immediately realize from an analysis of the gap equation.

We would like to mention that the numerical evaluation of higher orders of the $1/N$ expansion is technically not beyond reach, along the lines traced by Ref. [13] and exploiting the more accurate results for the effective propagator obtained in the present paper.

Finally, as a consequence of the discussion of the previous sections, it should be clear that, in the class of models with next-to-nearest neighbor interactions, it is always possible to find a choice of Hamiltonian parameters such that improvement becomes possible for arbitrary values of N.

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